MATH 2850: SUBSTITUTIONS - BERNOULLI EQUATIONS

DEFINITION: A **Bernoulli** equation is an ODE of the form: $y' + p(x)y = f(x)y^r$ for $r \neq 0, 1$.

Assuming $y \neq 0$, we may divide through by y^r to obtain: $y^{-r}\,y' + p(x)y^{1-r} = f(x)$.

Let $u = y^{1-r}$. The chain rule gives: $u' = (1-r)y^{-r}\,y'$ or $y^{-r}\,y' = \dfrac{1}{1-r}\,u'$.

Substituting into the original DE, we get: $\dfrac{1}{1-r}\,u' + p(x)u = f(x)$ which is a linear ODE in the variable u.

EXAMPLE: Consider the IVP: $y' + 2y = x\sqrt{y}$, $y(0) = 9$.

- Use the EUT to prove this IVP has a unique solution.

- Find the explicit solution to the IVP.

Ans: $y = \dfrac{1}{4}\left(x - 1 + 7e^{-x}\right)^2$

FUN FACT: WolframAlpha gives *two* solutions to this IVP even though there is just one . . .

EXAMPLE: Find an explicit family of solutions to : $x\,y' + x^3y^2 = 2y$. Assume $x > 0$.

Ans: $y = \dfrac{5x^2}{x^5 + 5C}$

EXAMPLE: Show the Logistic Equation: $P'(t) = k\,P(t)\,(L - P(t))$ is a Bernoulli Equation.

HOMEWORK: pg. 68: 1-17 odd, 23, 27

SUBSTITUTIONS - HOMOGENEOUS EQUATIONS

DEFINITION: A function $f(x, y)$ is said to be **homogeneous of degree** n if $f(tx, ty) = t^n f(x, y)$.

Homogeneous functions have their powers 'equally dispersed.'

EXAMPLE: Which of the following functions are homogeneous? If so, what is the degree?

- $f(x, y) = 4xy^2 + x^3 - 2x^2y$

Ans: homogeneous degree 3.

- $f(x, y) = 4y^2 + x^3 - 2x^2y$

Ans: not homogeneous

- $f(x, y) = \dfrac{6xy}{x^2 + y^2}$

Ans: homogeneous degree 0.

NOTE: If $f(x, y)$ is homogeneous of degree 0, then: $f(x, y) = f\left(x \cdot 1, x \cdot \dfrac{y}{x}\right) = x^0 f\left(1, \dfrac{y}{x}\right) = F\left(\dfrac{y}{x}\right)$.

EXAMPLE: Rewrite $f(x, y) = \dfrac{6xy}{x^2 + y^2}$ as a function $F\left(\dfrac{y}{x}\right)$.

DEFINITION: A differential equation $y' = f(x, y)$ is said to be **homogeneous** if f is homogeneous of degree 0.

Hence, a homogeneous DE has the form: $y' = F\left(\frac{y}{x}\right)$.

NOTE: This is another (different) definition of what it means for a DE to be 'homogeneous.' Context is key!

To solve a homogeneous ODE, we make the substitution $u = \frac{y}{x}$ or $y = ux$. Then $y' = u'x + u$. Hence:

$$y' = F\left(\frac{y}{x}\right) \implies u'x + u = F(u) \implies u'x = F(u) - u \implies \frac{1}{F(u) - u}\,du = \frac{1}{x}\,dx$$

In other words, we can transform homogeneous ODEs into separable ODEs.

EXAMPLE: Find an implicit one-parameter family of solutions to: $y' = \frac{y^2 - x^2}{2xy - x^2}$ and check your answer.

Ans: $x^2 + y^2 - xy - Cx = 0$ or $x + \frac{y^2}{x} - y = C$

EXAMPLE: Consider the IVP: $x^2 y' = 2x^2 + y^2 + 4xy$, $y(1) = 1$.

- Use Picard's EUT to show this IVP has a unique solution.

- Find the explicit solution to the IVP along with its interval of validity.

Ans: $y = \dfrac{3x - 4x^2}{2x - 3}$, $\left(0, \dfrac{3}{2}\right)$.

HOMEWORK: pg. 70: 25, 28-34 all